THE ASSOCIATED EXAMINING BOARD

General Certificate of Education

Advanced Level

June Examination, 1991

MATHEMATICS—PURE

Mathematics Paper 2 **A/MATH/2**
(632/2)

Thursday, 13 June, 1.30 p.m. to 4.30 p.m.

3 hours allowed

In addition to this paper you will require:

1. a 16-page answer booklet;
2. a booklet of formulae;
3. A4 squared paper (metric).

You may use an Electronic Calculator and/or Mathematical Tables.

Answer **seven** *questions only.*

All questions carry equal marks.

Unless stated otherwise, formulae may be quoted, without proof, from the booklet.

All necessary working should be shown and any numerical expression being evaluated by calculator or mathematical tables must be clearly stated; otherwise marks for method may be lost.

The **final** *answer to questions requiring the use of tables or calculators should normally be given correct to three significant figures.*

Mark allocations are shown in brackets.

1. (a) Prove, by mathematical induction, that for all positive integers n,

$$\sum_{r=1}^{n} r(r+1)\left(\tfrac{1}{2}\right)^{r-1} = 16 - (n^2 + 5n + 8)\left(\tfrac{1}{2}\right)^{n-1}.$$ *(6 marks)*

(b) Given that $0 < x < 1$, write down the sum of the infinite series

$$x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots + \frac{x^r}{r} + \ldots.$$ *(1 mark)*

By integrating this series term by term, show that

$$\frac{x^2}{1.2} + \frac{x^3}{2.3} + \frac{x^4}{3.4} + \ldots + \frac{x^{r+1}}{r(r+1)} + \ldots = x + (1-x)\ln(1-x).$$

(6 marks)

Hence, or otherwise, find the sum of the infinite series

$$\frac{1}{1.2} + \frac{1}{2.3}\left(\frac{1}{2}\right) + \frac{1}{3.4}\left(\frac{1}{2}\right)^2 + \ldots + \frac{1}{r(r+1)}\left(\frac{1}{2}\right)^{r-1} + \ldots.$$

(2 marks)

2. Sketch the curve with equation $9y^2 = x(3-x)^2$. *(2 marks)*

Show that, for $x \neq 3$,

$$4x\left[1 + \left(\frac{dy}{dx}\right)^2\right] = (x+1)^2.$$ *(4 marks)*

Hence prove that the length of the arc of the curve above the x-axis from $x = 1$ to $x = 2$ is $\frac{1}{3}(5\sqrt{2} - 4)$. *(3 marks)*

The line $x = a$ divides the loop of the curve into two regions of equal area. Show that

$$5a^{\frac{3}{2}} - a^{\frac{5}{2}} = 3\sqrt{3}.$$ *(3 marks)*

By using $a = 1.2$ as a first approximation to the root of this equation, use the Newton-Raphson method once to obtain a second approximation to the root, giving your answer to three significant figures. *(3 marks)*

3. The lines l_1 and l_2 have equations $\mathbf{r} = (3\mathbf{i} + \mathbf{j} - \mathbf{k}) + \alpha(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k})$ and $\mathbf{r} = (2\mathbf{i} + 5\mathbf{j}) + \beta(\mathbf{i} - \mathbf{j} + \mathbf{k})$ respectively.

(a) Prove that l_1 and l_2 intersect and find the position vector of P, the point of intersection. *(3 marks)*

(b) Determine the equation of the plane π containing l_1 and l_2, giving your answer in the form $\mathbf{r} \cdot \mathbf{n} = d$. *(4 marks)*

(c) The point Q has position vector $\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. The point R is such that QR is perpendicular to π and $QR = 2PQ$. Determine

(i) the position vectors of both possible positions of R. *(4 marks)*

(ii) the area of the triangle PQR, giving your answer to three significant figures. *(4 marks)*

4. The two points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ lie on the parabola with equation $y^2 = 4ax$. Show that the chord PQ has equation

$$(p+q)y = 2x + 2apq.$$ *(3 marks)*

Hence *deduce* the equations of the tangents at P and Q. *(2 marks)*

The tangents at P and Q meet at R. Prove that triangle PQR has area $\frac{1}{2}a^2|p-q|^3$. *(5 marks)*

The chord PQ moves so that the triangle PQR has constant area $4a^2$. Prove that PQ always touches the parabola with equation $y^2 = 4a(x-a)$. *(5 marks)*

[Turn over

5. Prove that $\sinh^{-1}x = \ln(x + \sqrt{(1 + x^2)})$. (4 marks)

(a) Given that $\exp(z) \equiv e^z$, show that $y = \exp(\sinh^{-1}x)$ satisfies the differential equation

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0. \quad (3\ marks)$$

Obtain the Maclaurin's expansion of y in ascending powers of x up to and including the term in x^4. (3 marks)

(b) Find the value of $\int_0^1 \sinh^{-1}x\,dx$, leaving your answer in terms of a natural logarithm. (5 marks)

6. Sketch the curve with equation $r = a\sin^2\theta$ where r, θ are polar coordinates and a is a positive constant. (3 marks)

Calculate the area enclosed by one loop of the curve. (5 marks)

Show that the curve may be written in cartesian coordinates as

$$(x^2 + y^2)^3 = a^2y^4. \quad (2\ marks)$$

The region enclosed by the upper loop is R. Determine the y-coordinate of the centroid of the solid formed by rotating R through π radians about the y-axis. (5 marks)

7. (a) Solve the equation $\left(\frac{z-1}{z+1}\right)^4 = -4$, giving your answers in the form $a + bi$, where a and b are real. (7 marks)

Show that, when the four roots are represented on an Argand diagram, they form the vertices of a trapezium and determine its area. (2 marks)

(b) The point P represents z on an Argand diagram. Given that

$$\operatorname{Arg}\left(\frac{z-3}{z+3}\right) = \frac{\pi}{4},$$

show that the locus of P is part of a circle. State the radius and centre of the circle and sketch the complete locus clearly. (6 marks)

8. Given that

$$I_n = \int \frac{x^n}{(a^2 + x^2)^{\frac{1}{2}}}\,dx,$$

where a is a constant, show that when $n \geqslant 2$,

$$nI_n = x^{n-1}(a^2 + x^2)^{\frac{1}{2}} - (n-1)a^2 I_{n-2}. \quad (5\ marks)$$

Hence,

(a) evaluate

$$\int_0^1 \frac{4x^4 + 7x^2 - 3}{(3 + x^2)^{\frac{1}{2}}}\,dx, \quad (3\ marks)$$

(b) solve the differential equation

$$(4 + x^2)\frac{dy}{dx} - xy = 3x^3(4 + x^2),$$

given that $y = 0$ when $x = 0$. (7 marks)

 [*Turn over*

9. Ann and Bob go out for the evening. The probability that they both go to the disco is $\frac{1}{12}$, whereas the probability that neither of them go to the disco is $\frac{2}{9}$.

(a) Calculate the probability that Ann or Bob, but not both of them go to the disco. *(3 marks)*

(b) "Ann goes to the disco" and "Bob goes to the disco" are independent events. Ann is more likely to go to the disco than Bob.

(i) Denoting by α, β the probabilities of Ann, Bob respectively going to the disco, write down the values of $\alpha + \beta$ and $\alpha\beta$, and by solving a quadratic equation, find the probability that Ann goes to the disco. *(6 marks)*

(ii) Calculate the probability that Ann goes to the disco, when it is known that just one of Ann and Bob goes to the disco. *(3 marks)*

(c) On one particular night there were 12 people, including Ann and Bob, on the dance floor. Three different people were chosen at random from the twelve and given a prize. What is the probability that neither Ann nor Bob received a prize? *(3 marks)*

10. Assuming a particular integral of the form kx^2e^{-2x}, find the solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 2e^{-2x},$$

satisfying $y = 1$ and $\frac{dy}{dx} = 0$ when $x = 0$. *(10 marks)*

Show that y is never negative.

Sketch the graph of y against x stating the coordinates of any turning points. *(5 marks)*